Announcements

1) Undergrad Research Showcase in March - deadline is 2/18

2) Presentation day is Wednesday 2/20

3) Covering sections 6.4 - 6.6

Infinite Series of Functions

We observed that if f is continuous on $[a,b]$, $\exists\ (P_n)_{n=1}^{\infty}$ polynomials, $P_n \to f$ uniformly on $[a,b]$.

So is f in some sense an "infinite" polynomial?

<u>Definition</u>: (convergence of series)

Let $(f_n)_{n=1}^{\infty}, f : A \to \mathbb{R}$ with $A \subseteq \mathbb{R}$. Consider, for all $x \in A$, the partial sums

$$S_k(x) = \sum_{n=1}^{k} f_n(x).$$

We say

$$\boxed{f(x) = \sum_{n=1}^{\infty} f_n(x)}$$

if for each $x \in A$,

$$\lim_{k \to \infty} S_k(x) = f(x), \text{ i.e.,}$$

$S_k \to f$ pointwise.

We can ask whether the convergence is uniform on A.

Example 1: (power series)

Choose $c \in \mathbb{R}$ and for a sequence $(a_n)_{n=0}^{\infty}$ of real numbers, let

$$f_n(x) = a_n (x-c)^n.$$

Then we write

$$\boxed{f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n}$$

is a power series.

The series certainly converges at $x = c$ (the center of the series). We will assume in what follows that $c = 0$; statements readily translate to $c \neq 0$.

Q1: Are there any other points at which the series converges?

Q2: If f is continuous on $A \subseteq \mathbb{R}$, does

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

for some sequence $(a_n)_{n=0}^{\infty}$?

(For $x \in A$, pointwise convergence) We'd like A to be an interval.

Propositions

1) Continuity: If $(f_n)_{n=1}^{\infty}$ and $f: A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and f_n is continuous on A $\forall n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} f_n$ converges to f uniformly, then f is continuous.

2) Differentiability:

Suppose $(f_n)_{n=1}^{\infty} : [a,b] \to \mathbb{R}$ is differentiable $\forall\, n \in \mathbb{N}$ and $x \in [a,b]$. Suppose

$$\sum_{n=1}^{\infty} f_n'(x) = g(x) \text{ uniformly}$$

on $[a,b]$ and $\exists\, x_0 \in [a,b]$,

$$\sum_{n=1}^{\infty} f_n(x_0) \text{ converges. Then}$$

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \text{ on } [a,b] \text{ and}$$

$$f'(x) = g(x) \text{ on } [a,b].$$

3) Uniformly Cauchy:

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \text{ uniformly}$$

on $A \subseteq \mathbb{R}$ if and only if

$(S_k)_{k=1}^{\infty}$, the sequence

of partial sums, is uniformly

Cauchy.

All three propositions follow immediately from the analagous results for <u>sequences</u> of functions by considering the sequence of partial sums $(S_k)_{k=1}^{\infty}$.

Theorem: (Weierstrass M-test)

Consider $(f_n)_{n=1}^{\infty} : A \to \mathbb{R}$

where $A \subseteq \mathbb{R}$. Suppose,

$\forall\ x \in A$, that $\exists\ (M_n)_{n=1}^{\infty}$

a sequence of real numbers

with $\boxed{|f_n(x)| \leq |M_n|}$ $\forall\ x \in A$

and each $n \in \mathbb{N}$. If

$\sum_{n=1}^{\infty} |M_n|$ converges, then $\sum_{n=1}^{\infty} f_n(x)$

converges uniformly on A.

proof: Examine the partial sums

$$\sum_{n=1}^{k} f_n = S_k .$$

Show that $(S_k)_{k=1}^{\infty}$ is uniformly Cauchy.

Then we'll be done by our proposition 3).

Suppose $k > m$.

$$\left| S_k(x) - S_m(x) \right|$$

$$= \left| \sum_{n=1}^{k} f_n(x) - \sum_{n=1}^{m} f_n(x) \right|$$

$$= \left| \sum_{n=m+1}^{k} f_n(x) \right|$$

$$\leq \sum_{n=m+1}^{k} \left| f_n(x) \right|$$

$$\leq \sum_{n=m+1}^{k} \left| M_n \right|$$

We know that

$$\sum_{n=1}^{\infty} |M_n| \text{ converges,}$$

so the sequence of partial sums $(T_k)_{k=1}^{\infty}$ is Cauchy. Therefore,

$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N}$ with

$$|T_k - T_m| < \varepsilon$$

$\forall\, k, m \geq N$.

But if $k > m$,

$$|\overline{T_k} - \overline{T_m}|$$

$$= \overline{T_k} - \overline{T_m}$$

$$= \sum_{n=m+1}^{k} |M_n| < \varepsilon \quad \forall k, m \geq N.$$

Then for all such k and m,

$$|S_k - S_m| \leq \sum_{n=m+1}^{k} |M_n| < \varepsilon.$$

Then $(S_k)_{k=1}^{\infty}$ is uniformly Cauchy on A, hence converges uniformly on A. $\square$

Example 2:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}, \quad \boxed{x \geq 0}$$

Where is the convergence uniform?

$$f(0) = \sum_{n=1}^{\infty} 1$$ which does not converge!

Suppose we consider $\delta = 1$. If $x > 1$, then we want to use the Weierstrass M-test.

$$f_n(x) = \frac{1}{1+n^2x} < \frac{1}{1+n^2} = M_n.$$

since $x > 1 \Rightarrow 1+n^2x > 1+n^2$.

At $x=1$, we have equality.

Therefore, by the Weierstrass M-test, the convergence is uniform on $[1, \infty)$.

The same argument with any $\delta > 0$ and $M_n = \frac{1}{1+n^2\delta}$ shows that the convergence is uniform on $[\delta, \infty)$.

Remember

$$\sum_{n=1}^{\infty} \frac{1}{1+n^{25}} \quad \text{converges}$$

since it is essentially

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \text{ which}$$

converges (comparison or limit comparison).